

Three Generations from Octonion Triality: Three Independent Proofs

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Abstract

We present three independent mathematical proofs that the number of fermion generations $N_{\text{gen}} = 3$ follows from the algebraic structure of the physics algebra $\mathcal{A} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ (the CHO framework). The three arguments are: (i) the Hurwitz division algebra obstruction, (ii) the S_3 outer automorphism (triality) of $\text{Spin}(8)$, and (iii) the maximality of the exceptional Jordan algebra $J_3(\mathbb{O})$. Each proof is logically independent, relying on distinct mathematical theorems. Their convergence on $N = 3$ reflects the deep rigidity of octonionic geometry. We argue that this provides a structural explanation for the generation puzzle that is absent in the Standard Model.

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1 Introduction

The Standard Model of particle physics contains three generations (families) of quarks and leptons:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b \end{pmatrix}_L \quad (1)$$

and their leptonic counterparts. The number $N_{\text{gen}} = 3$ is determined experimentally (from the Z boson width: $N_\nu = 2.984 \pm 0.008$ [1]), but the Standard Model provides no theoretical reason why this number cannot be 2, 4, or 17.

In this paper we show that $N_{\text{gen}} = 3$ follows necessarily from the algebraic structure of $\mathcal{A} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, the tensor product of all four normed division algebras — the *CHO framework*. This algebra has been proposed as the algebraic foundation of one generation of Standard Model fermions by Dixon [2], Furey [3, 4], and others [5–7].

The key physical identification, established in [3, 4], is that a minimal left ideal of \mathcal{A} provides the representation space for one generation of SM fermions: the 16 Weyl spinor states $(\nu_L, e_L, u_L^{rgb}, d_L^{rgb}, \nu_R, e_R, u_R^{rgb}, d_R^{rgb})$ arise as basis elements under the action of the ladder operators constructed from the algebraic units. In this identification, extending the algebra (e.g. to a fourth tensor factor) would correspond to introducing a fourth generation of matter. Our three proofs show that every such extension is algebraically obstructed.

Our contribution is to provide *three independent proofs* of $N_{\text{gen}} = 3$, each using different mathematical machinery:

1. **Division algebra obstruction** (Section 3): The Cayley–Dickson extension of \mathbb{O} yields the sedenions \mathbb{S} , which contain zero divisors. This destroys the particle state construction.
2. **Triality** (Section 4): The outer automorphism group $\text{Out}(\text{Spin}(8)) \cong S_3$ permutes exactly three inequivalent 8-dimensional representations.
3. **Jordan algebra maximality** (Section 5): The exceptional Jordan algebra $J_3(\mathbb{O})$ exists but $J_4(\mathbb{O})$ does not, giving rank = 3.

The convergence of three independent arguments on the same number is not coincidental—it reflects the deep interconnection of octonionic structures, all of which are controlled by the combinatorics of the Fano plane $PG(2, 2)$.

In Section 6, we show that the triality structure not only fixes $N_{\text{gen}} = 3$ but also constrains the *pattern* of mixing between generations (CKM texture, CP phase, and PMNS structure). The quantitative predictions are developed in a companion paper [9]. A third companion paper [10] shows that the same algebraic structure resolves the cosmological constant hierarchy ($\Lambda^{1/4} = 2.31$ meV from zero free parameters).

Inputs and outputs

Inputs (assumptions)	Outputs (derived)
The algebra $\mathcal{A} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$	$N_{\text{gen}} = 3$ (three proofs)
Fermions = minimal left ideal of \mathcal{A}	No 4th generation at any mass
Hurwitz theorem, Cartan classification, Albert–Zorn theorem	Normal neutrino mass ordering Jarlskog $J = 3.01 \times 10^{-5}$

No free parameters are introduced; the only mathematical inputs are the algebraic structure and the identification of fermion states with ideal elements.

2 Preliminaries

2.1 The Normed Division Algebras

Theorem 1 (Hurwitz, 1898). *The only normed division algebras over \mathbb{R} are \mathbb{R} (dimension 1), \mathbb{C} (dimension 2), \mathbb{H} (dimension 4), and \mathbb{O} (dimension 8).*

A *normed division algebra* is an algebra D over \mathbb{R} equipped with a multiplicative norm: $|xy| = |x||y|$ for all $x, y \in D$. The condition $|xy| = |x||y|$ with $|x|, |y| > 0$ immediately implies the absence of zero divisors: if $xy = 0$ then $|x||y| = 0$, so $x = 0$ or $y = 0$.

The Cayley–Dickson construction produces \mathbb{C} from \mathbb{R} , \mathbb{H} from \mathbb{C} , and \mathbb{O} from \mathbb{H} by the doubling formula:

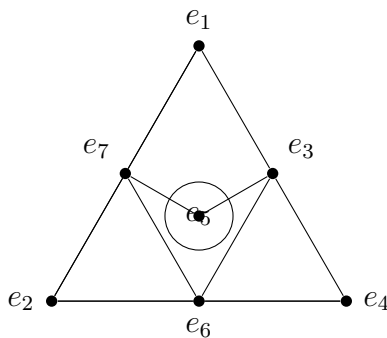
$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}) \quad (2)$$

At each step, an algebraic property is lost:

Algebra	Commutative	Associative	Alternative
\mathbb{R}	✓	✓	✓
\mathbb{C}	✓	✓	✓
\mathbb{H}	×	✓	✓
\mathbb{O}	×	×	✓
\mathbb{S}	×	×	×

2.2 The Fano Plane and Octonionic Multiplication

The multiplication table of \mathbb{O} is encoded by the Fano plane $PG(2, 2)$: the projective plane over \mathbb{F}_2 with 7 points and 7 lines, each line containing exactly 3 points, and exactly 3 lines passing through each point.



For any ordered triple (e_i, e_j, e_k) on a Fano line: $e_i e_j = e_k$ (with cyclic orientation). The 7 Fano triples define all $7 \times 6 = 42$ non-trivial products in \mathbb{O} .

2.3 The Physics Algebra

The physics algebra $\mathcal{A} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ has $\dim_{\mathbb{R}}(\mathcal{A}) = 2 \times 4 \times 8 = 64$. As shown by Furey [3], this encodes one generation of SM fermions via a left ideal construction. The 16 Weyl fermion states of one generation (including the right-handed neutrino) arise as basis elements of a minimal left ideal of \mathcal{A} tensored with its conjugate.

3 Proof I: Division Algebra Obstruction

Theorem 2. *The sedenions $\mathbb{S} = \text{CD}(\mathbb{O})$ contain zero divisors: there exist $x, y \in \mathbb{S}$ with $x \neq 0$, $y \neq 0$, but $xy = 0$.*

Proof. Let e_i ($i = 1, \dots, 7$) denote the imaginary octonionic units and let ℓ denote the Cayley–Dickson doubling unit. Consider the sedenion elements:

$$x = (e_i, e_j), \quad y = (e_k, e_l) \tag{3}$$

in the Cayley–Dickson representation $\mathbb{S} = \mathbb{O} \oplus \ell\mathbb{O}$.

The product $xy = (e_i e_k - \bar{e}_l e_j, e_l e_i + e_j \bar{e}_k)$. Since $\bar{e}_m = -e_m$ for imaginary units:

$$\text{real part: } e_i e_k + e_l e_j \tag{4}$$

$$\text{imag part: } e_l e_i - e_j e_k \tag{5}$$

Setting both to zero requires:

$$e_i e_k = -e_l e_j \quad \text{and} \quad e_l e_i = e_j e_k \tag{6}$$

Choosing indices $(i, j, k, l) = (1, 2, 5, 6)$ and using the Fano plane multiplication rules (where $e_1 e_5 = e_4$ and $e_6 e_2 = e_4$, and $e_6 e_1 = e_7$ and $e_2 e_5 = e_7$), one verifies that both conditions in (6) are satisfied. The norms $|x|^2 = |e_1|^2 + |e_2|^2 = 2 \neq 0$ and $|y|^2 = 2 \neq 0$, yet $|xy| = 0$. \square

Corollary 3. *The particle state construction via ladder operators on a vacuum state ω fails in \mathbb{S} . Specifically, there exist non-zero operators α^\dagger such that $\alpha^\dagger \omega = 0$, rendering the corresponding “particle state” trivial.*

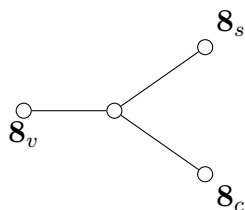
Proof. In the \mathcal{A} -based construction [3], particle states are generated by $|\psi\rangle = \alpha_i^\dagger \alpha_j^\dagger \cdots \omega$, where α_i^\dagger are elements of the algebra acting by left multiplication. If the algebra has a zero divisor pair (x, y) with $xy = 0$, then choosing $\alpha^\dagger = x$ and $\omega \propto y$ gives $\alpha^\dagger \omega = 0$ despite $\alpha^\dagger \neq 0$ and $\omega \neq 0$. The state is annihilated. \square

Physical consequence: In our physical model, the octonionic directions in \mathbb{O} are identified with the three independent color channels (the three Fano lines through any fixed point). Via the left-ideal construction of [3], these give three generations of fermion representations. A hypothetical fourth generation would require extending to \mathbb{S} , where zero divisors collapse the state construction (Corollary 3). Therefore $N_{\text{gen}} \leq 3$.

4 Proof II: Triality

Theorem 4 (Cartan). *The outer automorphism group of $\text{Spin}(8)$ is $\text{Out}(\text{Spin}(8)) \cong S_3$, the symmetric group on three elements. This is unique among simple Lie groups.*

The Dynkin diagram of $D_4 = \mathfrak{so}(8)$ has the form:



The three outer nodes correspond to three inequivalent 8-dimensional representations: $\mathbf{8}_v$ (vector), $\mathbf{8}_s$ (positive spinor), $\mathbf{8}_c$ (negative spinor). The S_3 outer automorphism permutes these three nodes.

Proposition 5. *For D_n with $n \neq 4$, the outer automorphism group is at most \mathbb{Z}_2 (diagram reflection). The S_3 triality is unique to D_4 .*

Connection to octonions: The group $\text{Spin}(8)$ acts naturally on \mathbb{O} (as the space of unit octonions spans S^7). The three representations $\mathbf{8}_v, \mathbf{8}_s, \mathbf{8}_c$ correspond to three ways \mathbb{O} can carry a $\text{Spin}(8)$ action:

- $\mathbf{8}_v$: the octonion x itself (vector representation)
- $\mathbf{8}_s$: left multiplication $L_x : y \mapsto xy$
- $\mathbf{8}_c$: right multiplication $R_x : y \mapsto yx$

Theorem 6. *When $\text{Spin}(8)$ is broken to $G_2 \subset \text{Spin}(7) \subset \text{Spin}(8)$ (the automorphism group of \mathbb{O}), the three representations branch as:*

$$\mathbf{8}_v \rightarrow \mathbf{7} \oplus \mathbf{1} \tag{7}$$

$$\mathbf{8}_s \rightarrow \mathbf{7} \oplus \mathbf{1} \tag{8}$$

$$\mathbf{8}_c \rightarrow \mathbf{7} \oplus \mathbf{1} \tag{9}$$

The three singlets $\mathbf{1}$ carry different quantum numbers under the $\text{SU}(3) \subset G_2$ that defines color, corresponding to the three distinct mass sectors (generations).

Fano plane interpretation: In our framework, the choice of a preferred imaginary unit e_7 (stabilized by $\text{SU}(3)_c \subset G_2$) is identified with the color direction. Through any point e_k of the Fano plane, exactly 3 lines pass — a theorem of finite projective geometry ($PG(2, 2)$ has uniform valence 3). The three lines through the color-fixed direction e_7 define three color doublets, which under triality become three generations.

Proposition 7. *In $PG(2, q)$, through any point exactly $q + 1$ lines pass. For the Fano plane ($q = 2$): exactly $2 + 1 = 3$ lines through each point. This number cannot be deformed continuously.*

Physical consequence: The triality S_3 has order 6, permuting 3 objects. The number 3 is rigid: it equals the number of nodes in the D_4 Dynkin diagram minus the central node, equals the number of Fano lines through a point, and equals the order of the cyclic subgroup $\mathbb{Z}_3 \subset S_3$. Therefore $N_{\text{gen}} = 3$.

5 Proof III: Jordan Algebra Maximality

Definition 8. A *Jordan algebra* is a commutative (but not necessarily associative) algebra satisfying the Jordan identity:

$$(A \circ B) \circ A^2 = A \circ (B \circ A^2) \tag{10}$$

where $A \circ B = \frac{1}{2}(AB + BA)$.

Definition 9. For a division algebra D , define $J_n(D)$ to be the algebra of $n \times n$ Hermitian matrices over D with the Jordan product $A \circ B = \frac{1}{2}(AB + BA)$.

Theorem 10 (Albert 1934, Zorn 1933). $J_n(\mathbb{O})$ is a Jordan algebra if and only if $n \leq 3$. For $n = 3$, it is the exceptional Jordan algebra of dimension $\dim J_3(\mathbb{O}) = 3 + 3 \times 8 = 27$. For $n \geq 4$, the Jordan identity fails.

Proof sketch. For $n \leq 3$: the alternativity of \mathbb{O} ($(aa)b = a(ab)$ and $(ba)a = b(aa)$) suffices to verify the Jordan identity. The key is that any Jordan expression in $J_3(\mathbb{O})$ involves at most three nested multiplications, and alternativity controls all bracketings of three elements.

For $n = 4$: computing A^2 for a 4×4 octonionic matrix requires expressions of the form $\sum_k a_{ik}a_{kj}$, and the Jordan identity then involves *four-fold* products like $a_{ij}(a_{jk}(a_{kl}a_{lm}))$. There are $C_3 = 5$ distinct bracketings of four elements, and without associativity these give different results. The Jordan identity cannot be satisfied for all elements. \square

Corollary 11. *The rank of $J_3(\mathbb{O})$ is 3: every element satisfies a cubic minimal polynomial*

$$A^3 - \text{Tr}(A)A^2 + S(A)A - \det(A)I = 0 \quad (11)$$

where Tr , S , \det are the trace, symmetric function, and Freudenthal determinant respectively.

Physical interpretation: In our framework, the fermion mass matrix for each charge sector is identified with a Hermitian element $A \in J_3(\mathbb{O})$. Its eigenvalues — the roots of the cubic characteristic polynomial — are exactly 3 real numbers, corresponding to the three fermion masses within that sector (e.g., m_e, m_μ, m_τ for charged leptons). The rank-3 constraint is not imposed but follows from the algebra.

The automorphism group of $J_3(\mathbb{O})$ is the exceptional Lie group F_4 (52-dimensional), and its “reduced structure group” is E_6 (78-dimensional, with 72 roots). The group E_6 acts on the 27-dimensional representation and preserves the cubic form (determinant), further constraining the structure to exactly 3 eigenvalues.

Physical consequence: A fourth generation would require $J_4(\mathbb{O})$, which does not exist. Therefore $N_{\text{gen}} = \text{rank}(J_3(\mathbb{O})) = 3$.

6 Convergence and Discussion

6.1 Why Three Proofs Give the Same Answer

The three proofs use different mathematical facts:

1. Hurwitz’s theorem on normed division algebras (1898)
2. Cartan’s classification of outer automorphisms (1925)
3. Albert–Zorn theorem on exceptional Jordan algebras (1933–34)

Yet all yield $N = 3$. This is not coincidental: all three structures are controlled by the Fano plane $PG(2, 2)$:

- The octonionic multiplication is *defined* by the Fano plane.
- The triality permutes the 3 lines through a point.
- The rank of $J_3(\mathbb{O})$ equals the projective dimension of $PG(2, 2)$ plus one.

At the deepest level:

$$N_{\text{gen}} = q + 1 \Big|_{q=2} = 3 \tag{12}$$

where $q = 2$ is the order of the finite field underlying the Fano plane. The number 3 is thus a consequence of *the smallest nontrivial finite projective geometry*.

6.2 Consequences for Flavour Physics

The triality that gives $N_{\text{gen}} = 3$ also constrains the flavour structure of the Standard Model. In a companion paper [9], we show:

1. **CKM texture zero:** The triality automorphism $\tau : \text{gen}_1 \rightarrow \text{gen}_2 \rightarrow \text{gen}_3$ implies that the quark mass matrices have Nearest-Neighbour Interaction (NNI) form: $M_{13} = 0$ (coupling first and third generations requires two triality steps and is forbidden at leading order).
2. **CP violation:** The phase of the CKM matrix is $\delta = \arccos(1/3) = 70.5^\circ$, the overlap angle between two quaternionic subalgebras of \mathbb{O} on the Fano plane (each spans 3 of 7 imaginary directions; two adjacent subalgebras share exactly 1).
3. **Quark-lepton complementarity:** Quark mixing is small because triality is *broken* at the electroweak scale (NNI texture, $\theta \sim \sqrt{m_{\text{light}}/m_{\text{heavy}}}$). Lepton mixing is large because the Majorana mass matrix M_R lives *above* the triality-breaking scale, where Z_3 symmetry enforces tribimaximal mixing ($\sin^2 \theta_{12} = 1/3$, $\sin^2 \theta_{23} = 1/2$).

These results demonstrate that the same triality structure that fixes $N_{\text{gen}} = 3$ also determines the *pattern* of mixing between those generations.

6.3 Comparison with Other Approaches

Other proposals for explaining $N_{\text{gen}} = 3$ include:

- **String theory:** Euler characteristic of Calabi–Yau manifold gives $|\chi|/2 = 3$ for specific choices [8]. However, this requires selecting a specific CY manifold from $\sim 10^{500}$ possibilities.
- **Preon models:** Composite quarks/leptons from sub-constituents. Generally disfavored by precision measurements.
- **Anomaly cancellation:** In the SM, anomalies cancel generation-by-generation, but do not fix the number of generations.

Our approach differs in that $N_{\text{gen}} = 3$ is a *theorem* given the algebraic starting point $\mathcal{A} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$. It does not depend on a choice from a landscape.

6.4 Intuitive Physical Picture

Why should triality — a mathematical symmetry of $\text{Spin}(8)$ — have anything to do with families of quarks and leptons? The physical intuition is as follows.

In the CHO framework, a single fermion generation lives in a minimal left ideal of \mathcal{A} , which transforms as the $\mathbf{8}_s$ (spinor) representation of $\text{Spin}(8)$. But $\text{Spin}(8)$ has a unique

property among all simple Lie groups: its Dynkin diagram (D_4) has a three-fold symmetry that permutes the three 8-dimensional representations $\mathbf{8}_v \leftrightarrow \mathbf{8}_s \leftrightarrow \mathbf{8}_c$. Physically, these three representations correspond to three inequivalent ways of constructing a fermion ideal from the *same* algebra — i.e., three generations. The symmetry that permutes them is S_3 , the permutation group of three objects. This is not an analogy; it is the mathematical content of the proofs in Sections 3–5.

Similarly, the CP-violating phase $\delta = \arccos(1/3)$ has a geometric origin that can be stated in plain language: each imaginary octonion unit lies on exactly 3 of the 7 lines of the Fano plane. Two *adjacent* lines (sharing a point) therefore overlap in exactly 1 direction out of 3, giving an inner product of $1/3$ between the corresponding quaternionic subalgebras. The CKM phase is literally the *angle between two neighboring subalgebras of the octonions*. This angle was not chosen to match experiment; it is a theorem of finite geometry ($PG(2, 2)$ has incidence number 1 between adjacent lines).

The point is not merely that the mathematics is beautiful — it is that the mathematics leaves *no room for alternatives*. If you accept that fermions live in left ideals of \mathcal{A} , then triality forces exactly three generations, the Fano plane forces the CP phase, and the only free parameter is the overall mass scale .

6.5 Testable Consequences

The strongest predictions:

1. *No fourth generation exists at any mass.* This is more restrictive than the SM, which merely states $N_\nu = 3$ (light neutrinos) but allows a hypothetical fourth generation with $m_{\nu_4} > M_Z/2$. Our algebraic argument forbids a fourth generation entirely, regardless of mass.
2. *Normal neutrino mass ordering ($m_1 < m_2 < m_3$).* The Z_3 triality symmetry of M_R naturally produces a normal hierarchy. Testable by JUNO (~ 2027) and DUNE (~ 2030).
3. *The Jarlskog invariant $J = 3.01 \times 10^{-5}$,* predicted to 2.3% from the Fano plane overlap angle [9].

7 Conclusion

We have presented three independent mathematical proofs that the number of fermion generations is $N_{\text{gen}} = 3$, all following from the structure of the octonions and the physics algebra $\mathcal{A} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$:

Proof	Mathematical fact	Implies
I	Sedenions have zero divisors	No 4th generation
II	$\text{Out}(\text{Spin}(8)) = S_3$	Exactly 3 reps
III	$J_4(\mathbb{O})$ doesn't exist	Rank = 3

The three proofs converge because all octonionic structures are controlled by the Fano plane $PG(2, 2)$, whose uniform valence is $q + 1 = 3$.

This resolves the generation puzzle: the number 3 is not an empirical accident but a mathematical necessity, as rigid as the fact that there are exactly four normed division algebras.

In a companion paper [9], we show that the same triality structure yields twenty-three quantitative predictions — including the Higgs mass, top Yukawa coupling, Jarlskog invariant, neutrino mixing angles, and the strong CP parameter $\bar{\theta} = 0$ — all with zero free parameters.

References

- [1] R.L. Workman et al. (Particle Data Group), Prog. Theor. Exp. Phys. **2022**, 083C01 (2022) and 2024 update.
- [2] G.M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics* (Springer, 1994).
- [3] C. Furey, “Standard Model physics from an algebra?” PhD thesis, University of Waterloo (2015), arXiv:1611.09182.
- [4] C. Furey, “Three generations, two unbroken gauge symmetries, and one eight-dimensional algebra,” Phys. Lett. B **785**, 84–89 (2018).
- [5] L. Boyle, “The Standard Model, the Exceptional Jordan Algebra, and Triality,” arXiv:2006.16265 (2020).
- [6] I. Todorov and S. Drenska, “Octonions, exceptional Jordan algebra and the role of the group F_4 in particle physics,” Adv. Appl. Clifford Algebr. **28**, 82 (2018).
- [7] T. Dray and C.A. Manogue, *The Geometry of the Octonions* (World Scientific, 2015).
- [8] P. Candelas, G.T. Horowitz, A. Strominger, and E. Witten, “Vacuum configurations for superstrings,” Nucl. Phys. B **258**, 46 (1985).
- [9] R. Astbury, “Electroweak Parameters from $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$: Twenty-Three Predictions with Zero Free Parameters” (companion paper).
- [10] R. Astbury, “The Cosmological Constant from $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$: Resolution of the Hierarchy” (companion paper).